Computing Endomorphism Rings of Abelian Varieties

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Isogenies and Abelian Varieties

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- If $\mathrm{End}(\mathscr{A}) \neq \mathrm{End}(\mathscr{A}')$, take a d-isogeny, and then...
- If $\mathrm{End}(\mathscr{A}) = \mathrm{End}(\mathscr{A}')$, use Pollard's rho (or a quantum computer).

Computing Endomorphism Rings (easy part)

Assume we can test whether $\mathscr{O} \subseteq \mathrm{End}(\mathscr{A})$...

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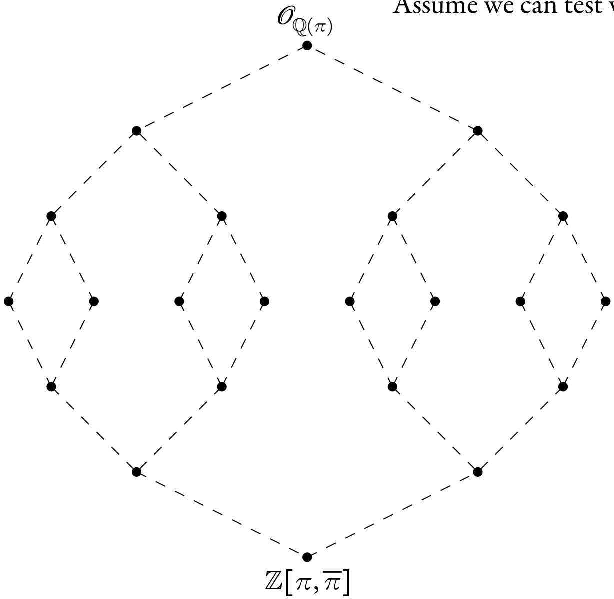

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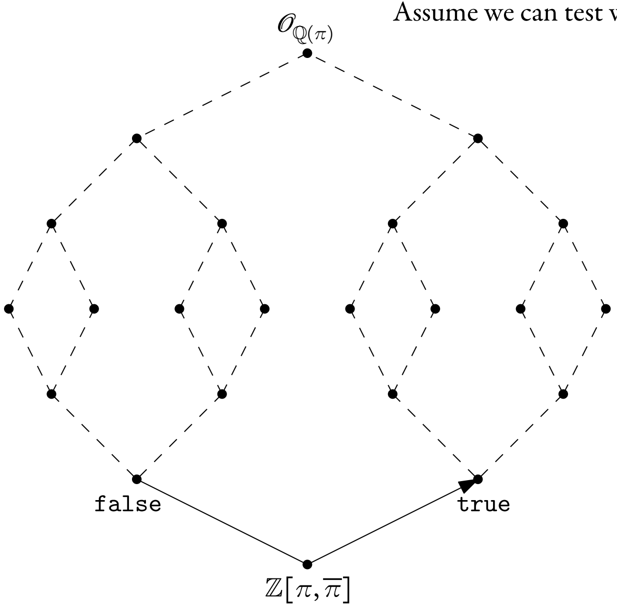

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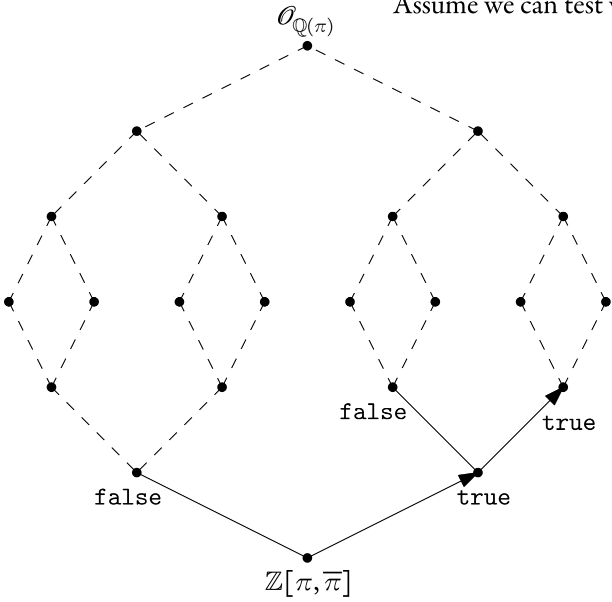

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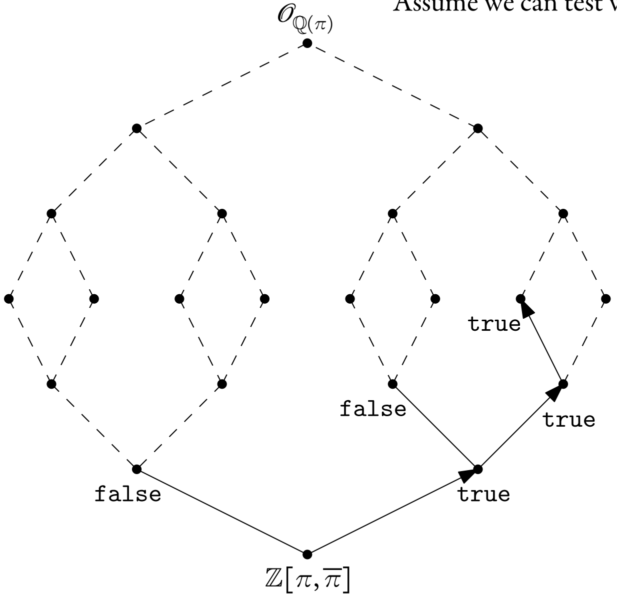

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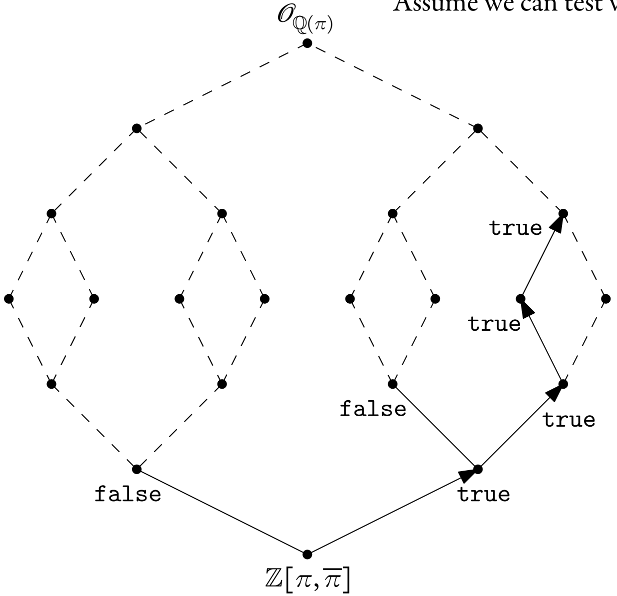

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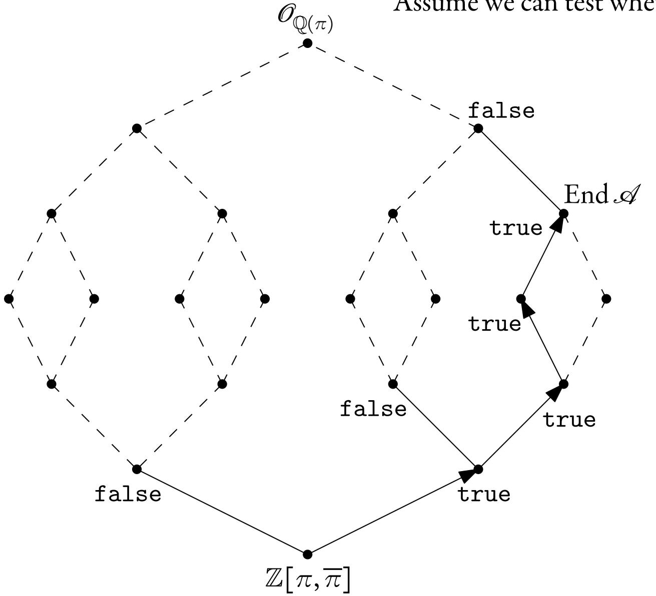

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Using the *horizontal structure*, we design a subexponential algorithm:

- fast and proven under GRH for $g = 1$;
- slower and relies on more heuristics for $g = 2$.

(Partly joint work with Drew Sutherland.)

Vertical vs. Horizontal

An ℓ-isogeny $\phi : \mathscr{A} \to \mathscr{A}'$ is:

- *vertical* if $\mathrm{End}(\mathscr{A}) \neq \mathrm{End}(\mathscr{A}')$
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Now, fix a base field $\mathbb{F}_q$, a conjugacy class for π, and a prime ℓ.

Right:

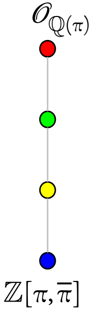
- $V = \{$orders containing $\mathbb{Z}[\pi, \overline{\pi}]\}$
- $E =$ inclusion

Left: (one connected component of)

- $V = \{$isomorphism classes of p.p. abelian varieties$\}$
- $E = \{\ell$-isogenies$\}$

Vertical Structure for $g = 1$

(Kohel's thesis)



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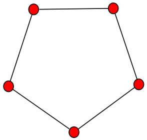

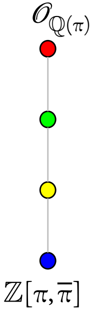

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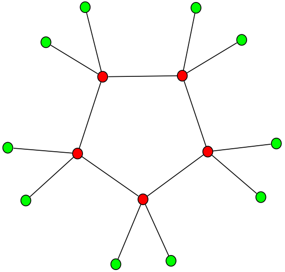

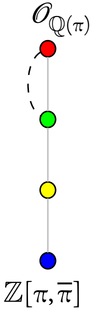

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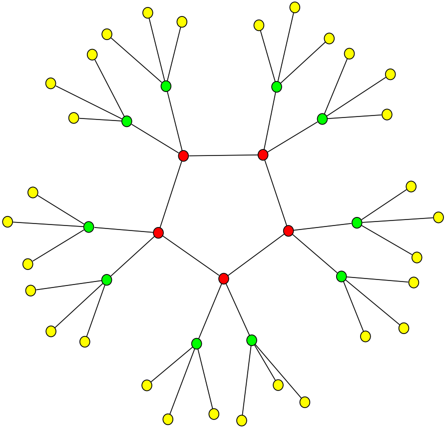

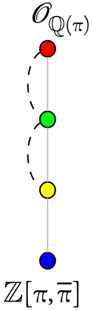

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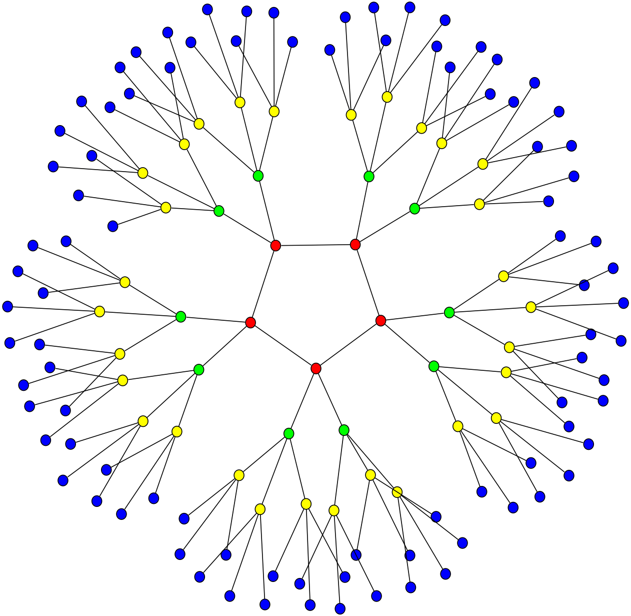

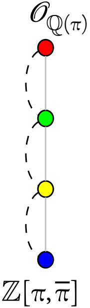

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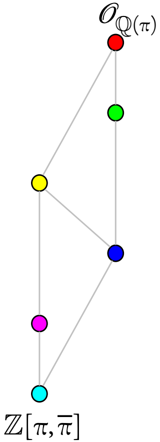

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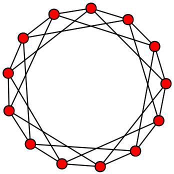

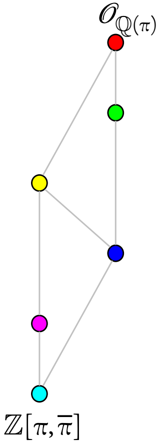

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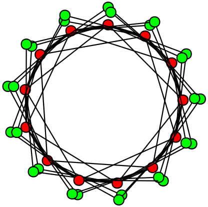
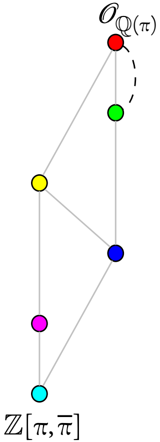
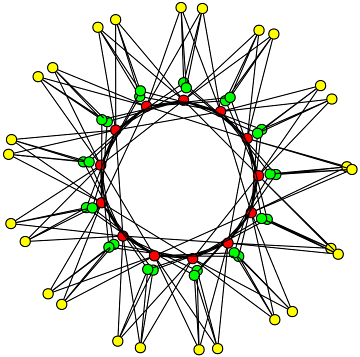
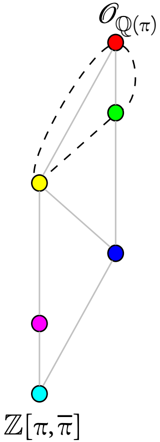




VERTICAL STRUCTURE FOR $g = 2$





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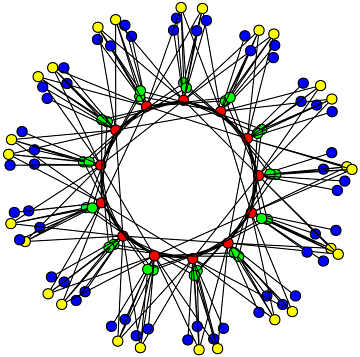

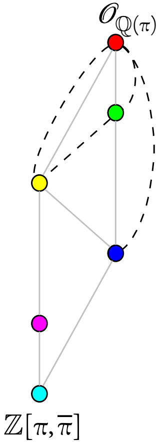

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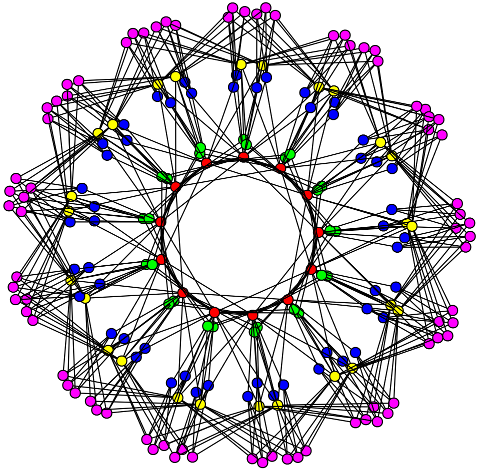

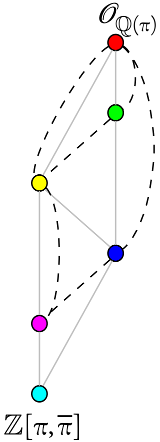

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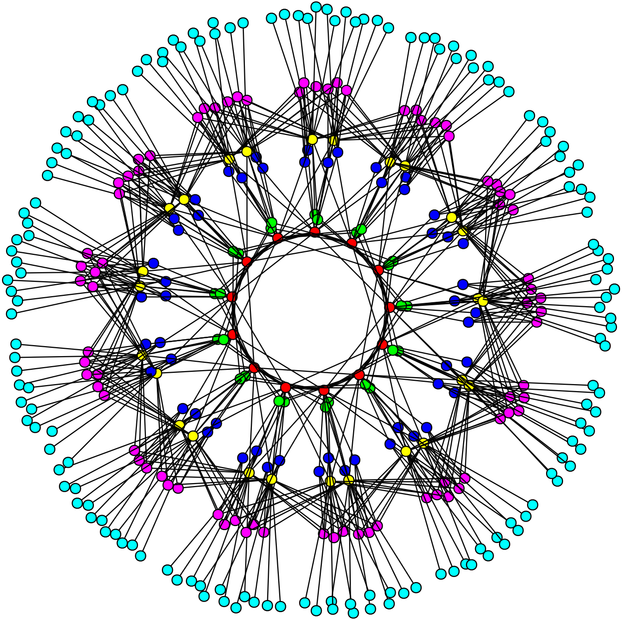

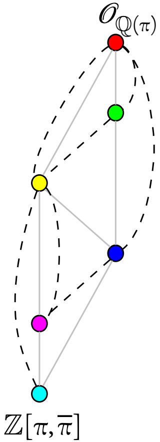

Complex Multiplication

Ideals $\mathfrak{a}$ such that $\mathfrak{a}\bar{\mathfrak{a}} = \ell\mathscr{O}$ act as ℓ-isogenies on $\{\mathscr{A} : \mathrm{End}(\mathscr{A}) \simeq \mathscr{O}\}$.
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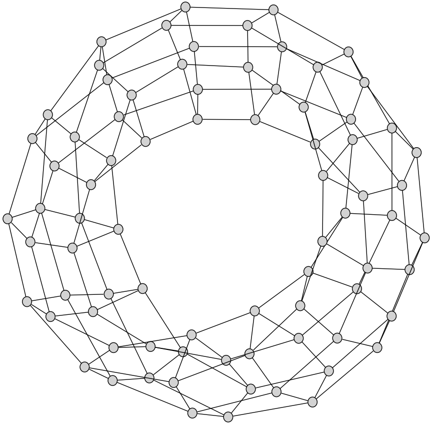

Example:
$(\mathfrak{p}\mathfrak{q})^{26} = 1$
$(\mathfrak{p}\bar{\mathfrak{q}})^{6} = 1$
$(\mathfrak{p}\mathfrak{q})^{13}(\mathfrak{p}\bar{\mathfrak{q}})^{3} = 1$

Probing Class Groups

$$\mathfrak{a}\ \mathfrak{b}\ \mathfrak{c} = 1 \in \mathrm{cl}(\mathscr{O}')$$

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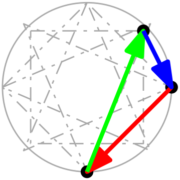

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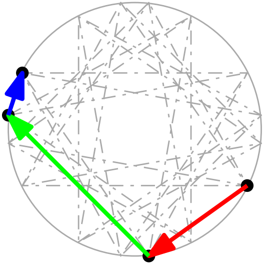

$\mathrm{End}(\mathscr{A}) \simeq \mathscr{O}$

Relations

Let $\mathfrak{P}$ be a generating set of ideals for $\mathrm{cl}(\mathbb{Z}[\pi, \overline{\pi}])$.

Define $\Lambda_{\mathscr{O}} = \{x \in \mathbb{Z}^{\mathfrak{P}} : \prod (\mathfrak{p}\mathscr{O})^{x_{\mathfrak{p}}} \text{ principal}\}$; thus $\mathbb{Z}^{\mathfrak{P}} / \Lambda_{\mathscr{O}} = \mathrm{cl}(\mathscr{O})$.

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- Ensure $\Lambda_{\mathcal{O}}$ determines $\mathcal{O}$. $\longrightarrow$ vertical methods

Finding Relations

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1. Let $\mathfrak{P} = \{\mathfrak{p} \text{ prime of norm} < N\}$
2. While true do:
3. Draw $x \in \{0, \ldots, h-1\}^{\mathfrak{P}}$ uniformly at random.
4. Let $y \leftarrow \prod \mathfrak{p}^{x_\mathfrak{p}}$.
5. Let $y' \leftarrow \mathrm{Reduce}(y)$.
6. If $y' = \prod \mathfrak{p}^{z_\mathfrak{p}}$ for some z, return $x - z$.

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Time: $L(\Delta)^\gamma + L(\Delta)^{1/(4\gamma)}$ when $N = L(\Delta)^\gamma$, $\Delta = \text{disc}(\mathcal{O})$.

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Find products of elements of $\mathfrak{P}$ that are principal in $\mathcal{O}$.

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Under GRH, for all $\varepsilon > 0$ there exists $c > 1$ such that for any order $\mathcal{O}$:
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This implies $\mathrm{diam}(\Lambda_{\mathcal{O}}) = o(\log^{4+\varepsilon}\Delta)$, from which we deduce
that *random* relations with small coefficients can be generated.

Restricting to ℓ-isogenies

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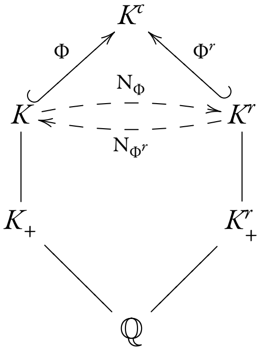

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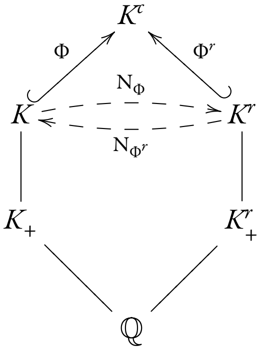

Practical Solution: Use BSGS.

Theoretical Results

Heuristics (only GRH needed for $g = 1$):

- GRH and smoothness of reduced ideals;
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Computing isogenies: ℓ^{4g}

Computing $\mathrm{End}(\mathscr{A})$ for an abelian variety $\mathscr{A}/\mathbb{F}_q$ takes time

$$L(q)^{g^{3/2}} \quad \text{for } g = 2$$

$$L(q)^{1/\sqrt{2}} \quad \text{for } g = 1 \quad \text{(faster isogenies, besides factoring)}$$

Practical Results for $g = 1$

Let $\mathscr{A}/\mathbb{F}_q$ be the elliptic curve $Y^2 = X^3 - 3X + c$ where

$$c = 66089717007102549448903693691119613107552207997068089804952 8$$
$$q = 160693804425899027555081234320605007554655094341590901447829 9$$

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Using further improvements for $g = 1$ yield the timings:

- four minutes to find relations;
- five minutes to evaluate the corresponding isogenies.

A typical relation was:

$$\mathfrak{p}_2^{1798}\mathfrak{p}_{23}^{3}\mathfrak{p}_{29}^{1}\mathfrak{p}_{37}^{2}\mathfrak{p}_{53}^{29}\mathfrak{p}_{137}^{1}\mathfrak{p}_{149}^{1}\mathfrak{p}_{233}^{1}\mathfrak{p}_{263}^{2}\mathfrak{p}_{547}^{1}$$

Practical Results for $g = 2$

Best Case: $\mathrm{Jac}\left(y^2 = 80742x^5 + 56078x^4 + 76952x^3 + 134685x^2 + 60828x + 119537\right)$ over $\mathbb{F}_{161983}$

$$[\mathcal{O}_{\mathbb{Q}(\pi)} : \mathbb{Z}[\pi, \overline{\pi}]] = 156799$$

The ideal $\mathfrak{p}_3^{115}$ is principal in $\mathcal{O}_{\mathbb{Q}(\pi)}$ but not in $\mathbb{Z}[\pi, \overline{\pi}]$.
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$$[\mathscr{O}_{\mathbb{Q}(\pi)} : \mathbb{Z}[\pi, \overline{\pi}]] = 13^2 \cdot 37 \cdot 79$$

Horizontal 3, 5, and 7-isogenies take 1, 3.5, and 5.5 seconds to compute.
Using $\mathfrak{p}_3^5 \mathfrak{p}_7^7 = 1$ and $\mathfrak{p}_5^{10} = 1$ suffices to conclude.

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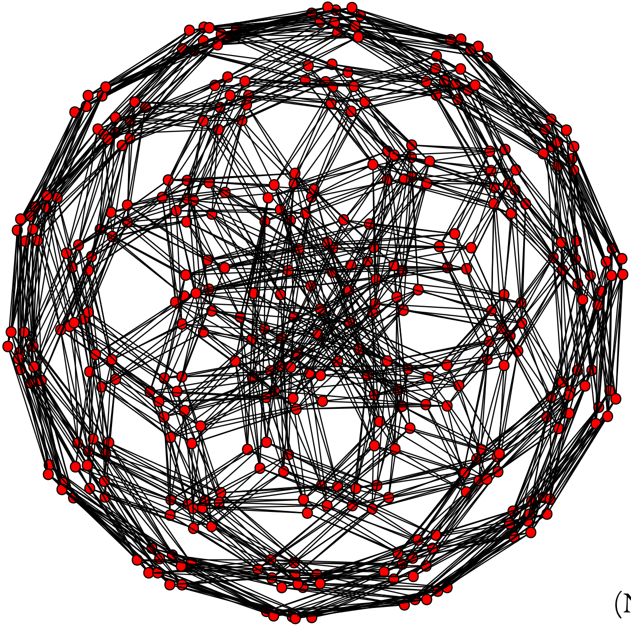
Worst Case: $[\mathcal{O}_{\mathbb{Q}(\pi)} : \mathbb{Z}[\pi, \overline{\pi}]] = 2 \cdot 3 \cdot 5$; slower than other methods.

NEXT YEAR: $g = 3$?!



(Not a chance.)