On the discrete logarithm problem in elliptic curves

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Some history

At ECC 2004 in Bochum, Pierrick Gaudry presented an index calculus algorithm for the ECDLP over extension fields:

**Heuristic claim** Let $n \in \mathbb{N}, n \geq 2$ be fixed. Then the ECDLP over fields of the form $\mathbb{F}_{q^n}$ can be solved in an expected time of

$$O(q^{\frac{2}{n}}).$$
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On the next day, I claimed:
Some history

**Claim.** There exists a randomized algorithm which takes as input a tuple \((q, n, E/\mathbb{F}_{q^n}, A, B)\), where \(q\) is a prime power, \(n\) a natural number, \(E/\mathbb{F}_{q^n}\) an elliptic curve and \(A, B \in E(\mathbb{F}_{q^n})\) with \(B \in \langle A \rangle\), which computes the DLP with respect to \(A\) and \(B\) and has the following property:

Let us fix \(a, b \in \mathbb{R}\) with \(0 < a < b\) and let us consider all instances with

\[
a \log_2(q) \leq n \leq b \log_2(q).
\]

Then restricted to these instances, the algorithm has an expected running time of

\[
O\left(2^{D \cdot (n \cdot \log_2(q))^{3/4}}\right) \text{ for } D = \frac{4b + \epsilon}{a^{3/4}}.
\]
Some history

And I continued ...
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Please note.

1. I do not have a complete proof of this statement.
2. The algorithm is not practical.
The good (and the bad) news

There is now a proven result:
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For fixed $a, b > 0$ and instances with

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The good (and the bad) news

- There is now a proven result: 
  For fixed $a, b > 0$ and instances with 
  \[ a \log(q)^{1/3} \leq n \leq b \log(b) \]
  we have an expected time of 
  \[ e^{O((\log(q^n))^{3/4})} \].

- The algorithm is still not practical.
A preliminary algorithm

Let an instance $\mathbb{E}/\mathbb{F}_{q^n}, A, B$ be given, $E$ in Weierstraß-Form. Let us for simplicity assume that $\#E(\mathbb{F}_{q^n})$ is prime. Let $k := \mathbb{F}_q$, $K := \mathbb{F}_{q^n}$, and let $x : \mathbb{E} \longrightarrow \mathbb{P}^1_K$ be as usual.
A preliminary algorithm

1. Determine $N := \#E(K)$. 
A preliminary algorithm

1. Determine $N := \#E(K)$.
2. Determine some $m \leq n$ and $c \leq n$.
3. Choose some $c$-dimensional $k$-vector subspace $U$ of $K$.
4. Define a so-called factor base

\[ \mathcal{F} := \{ P \in E(K) \mid x(P) \in U \} \]

Let $\mathcal{F} = \{ F_1, \ldots, F_k \}$.
A preliminary algorithm

5. For \( i = 1, \ldots, k + 1 \) do
   Repeat
   Choose \( \alpha_i, \beta_i \in \mathbb{Z}/N\mathbb{Z} \) uniformly randomly and try to determine a relation
   \[
P_1 + \cdots + P_m = \alpha_i A + \beta_i B
   \]
   with \( P_1, \ldots, P_m \in \mathcal{F} \).
   Until this was successful.
   Rewrite the relation as
   \[
   \sum_{j=1}^{k} r_{i,j} F_j = \alpha_i A + \beta_i B .
   \]
A preliminary algorithm

6. Determine some $\gamma \in (\mathbb{Z}/N\mathbb{Z})^{k+1} : \gamma R = 0, \gamma \neq 0$. 

We have 

$$(\sum_i \gamma_i \alpha_i) a + (\sum_i \gamma_i \beta_i) b = 0$$

and thus 

$$b = \frac{-\sum_i \gamma_i \alpha_i}{\sum_i \gamma_i \beta_i} a .$$
Relation generation

Given \( C (= \alpha A + \beta B) \in E(K) \), we want to find a relation

\[ P_1 + \cdots + P_m = C \]

with \( P_1, \ldots, P_m \in \mathcal{F} \).

For this we try to solve systems of multivariate polynomial equations over \( k \).
Relation generation

Idea. For \( P_1, \ldots, P_m \in E(K) \), the condition \( P_1 + \cdots + P_m = C \) can be expressed algebraically over \( K \).

We try to find relations by solving systems of polynomial equations over \( k \).

- The space of tuples \( (P_1, \ldots, P_m) \in F^m \) has \( mc \) degrees of freedom over \( k \).
- The space of points \( C \in E(K) \) has \( n \) degrees of freedom over \( k \).
Idea. For \( P_1, \ldots, P_m \in E(K) \), the condition \( P_1 + \cdots + P_m = C \) can be expressed algebraically over \( K \).

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Let \( \delta := mc - n \). Then for fixed \( C \) the relations / solutions \( (P_1, \ldots, P_m) \in \mathcal{F}^m \) with \( P_1 + \cdots + P_m = C \) vary in a \( \delta \)-dimensional space over \( k \).
Relation generation

**Idea.** For $P_1, \ldots, P_m \in E(K)$, the condition $P_1 + \cdots + P_m = C$ can be expressed algebraically over $K$.

We try to find relations by solving systems of polynomial equations over $k$.

- The space of tuples $(P_1, \ldots, P_m) \in \mathcal{F}^m$ has $mc$ degrees of freedom over $k$.
- The space of points $C \in E(K)$ has $n$ degrees of freedom over $k$.

$\implies$ Let $\delta := mc - n$. Then for fixed $C$ the relations / solutions $(P_1, \ldots, P_m) \in \mathcal{F}^m$ with $P_1 + \cdots + P_m = C$ vary in a $\delta$-dimensional space over $k$.

We want that $\delta = 0$ ...
1. Determine $N := \#E(K)$.

2. Determine some $m \leq n$, let $c := \lceil \frac{n}{m} \rceil$ and $\delta := mc - n$. We thus have $n = mc - \delta = (m - \delta) \cdot c + \delta \cdot (c - 1)$.

3. Choose some $c$-dimensional $k$-vector subspace $U$ of $K$ and some $c - 1$-dimensional $k$-vector subspace $U'$ of $U$.

4. Define a factor base

$$\mathcal{F} := \{ P \in E(K) \mid x(P) \in U \}$$

and also

$$\mathcal{F}' := \{ P \in E(K) \mid x(P) \in U' \}.$$

Let $\mathcal{F} = \{ F_1, F_2, \ldots, F_k \}$. 
A new preliminary algorithm

5. For $i = 1, \ldots, k + 1$ do
   Repeat
      Choose $\alpha_i, \beta_i \in \mathbb{Z}/N\mathbb{Z}$ uniformly randomly and try to determine a relation

      $$ P_1 + \cdots + P_m = \alpha_i A + \beta_i B $$

      with $P_1, \ldots, P_\delta \in \mathcal{F}', \quad P_\delta + 1, \ldots, P_m \in \mathcal{F}.$
   Until this was successful.
   Rewrite the relation as

   $$ \sum_{j=1}^{k} r_{i,j} F_j = \alpha_i A + \beta_i B. $$
6. Determine some $\gamma \in (\mathbb{Z}/N\mathbb{Z})^{k+1} : \gamma R = 0, \gamma \neq 0$.

We have

$$(\sum_i \gamma_i \alpha_i) a + (\sum_i \gamma_i \beta_i) b = 0$$

and thus

$$b = \frac{-\sum_i \gamma_i \alpha_i}{\sum_i \gamma_i \beta_i} a.$$
Decomposition

We need a procedure to compute relations or “decompositions”.

Input. $C \in E(K)$.

Output. A relation

$$P_1 + \cdots + P_m = C$$

with

$$P_1, \ldots, P_\delta \in \mathcal{F}', P_{\delta+1}, \ldots, P_m \in \mathcal{F},$$

that is,

$$x(P_1), \ldots, x(P_\delta) \in U', x(P_{\delta+1}), \ldots, x(P_m) \in U.$$
Decomposition

Let $P_1, \ldots, P_m \in E(K)$.

Equivalent are:

$P_1 + \cdots + P_m = C$
Decomposition

Let $P_1, \ldots, P_m \in E(K)$.

Equivalent are:

- $P_1 + \cdots + P_m = C$

- $(P_1) + \cdots + (P_m) + (\neg C) \sim (m + 1) \cdot O$
Decomposition

Let $P_1, \ldots, P_m \in E(K)$.

Equivalent are:

1. $P_1 + \cdots + P_m = C$
2. $(P_1) + \cdots + (P_m) + (C) \sim (m + 1) \cdot O$
3. $\exists f \in K(E)^*: (f) = (P_1) + \cdots + (P_m) + (C) - (m + 1) \cdot (O)$.
4. $\exists f \in L((m + 1) \cdot O - (C)) : (f) = (P_1) + \cdots + (P_m) + (C) - (m + 1) \cdot (O)$. 
Decomposition

Let \( P_1, \ldots, P_m \in E(K) \). Let \( P_1, \ldots, P_m, C, O \) be distinct.

Equivalent are:

1. \( P_1 + \cdots + P_m = C \)
2. \( (P_1) + \cdots + (P_m) + (\mathbf{C}) \sim (m + 1) \cdot O \)
3. \( \exists f \in K(E)^* : (f) = (P_1) + \cdots + (P_m) + (\mathbf{C}) - (m + 1) \cdot (O) \).
4. \( \exists f \in L((m + 1) \cdot O - (\mathbf{C}')) : 
   (f) = (P_1) + \cdots + (P_m) + (\mathbf{C}) - (m + 1) \cdot (O) \).
5. \( \exists f \in L((m + 1) \cdot O - (\mathbf{C}')) : \forall i = 1, \ldots, m : f(P_i) = 0 \).
Decomposition

Let $P_1, \ldots, P_m \in E(K)$. Let $P_1, \ldots, P_m, C, O$ be distinct. Equivalent are:

- $P_1 + \cdots + P_m = C$
- $(P_1) + \cdots + (P_m) + (-C) \sim (m + 1) \cdot O$
- $\exists f \in K(E)^* : (f) = (P_1) + \cdots + (P_m) + (-C) - (m+1) \cdot (O)$.
- $\exists f \in L((m + 1) \cdot O - (-C')) :$
  
  $(f) = (P_1) + \cdots + (P_m) + (-C) - (m + 1) \cdot (O)$.
- $\exists f \in L((m + 1) \cdot O - (-C')) : \forall i = 1, \ldots, m : f(P_i) = 0.$

Now: Choose a basis of $L((m + 1) \cdot O - (-C'))$, expand this over $k$, restrict $x(P_i)$ to $U$ or to $U'$...
Decompositions

Let $C, P_1, \ldots, P_m \in E(K)$. Let $P_1, \ldots, P_m, C, O$ be distinct. Let $b_1, \ldots, b_m$ be a basis of $L((m + 1) \cdot O - (-C))$.

Equivalent are:

1. $P_1 + \cdots + P_m = C$
2. $\exists \alpha_1, \ldots, \alpha_m \in K : \forall i = 1, \ldots, m : (\sum_\ell \alpha_\ell b_\ell)(P_i) = 0$

For varying $P_1, \ldots, P_m$, we have

- $2m$ variables for the $P_i$ and $m$ equations of degree 3
- $m - 1$ variables for the $\alpha_1, \ldots, \alpha_{m-1}$ and $m$ equations of low degree.

Over $k$, we have

- $nm + n$ variables and $nm$ equations for the $P_i$
- $nm - n$ variables and $nm$ additional equations.
Solving the systems

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In total: $2nm$ variables and $2nm$ equations of low degree over $k$.

$\implies$ We can expect that the system has 0-dimensional solution set.
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- $nm + n$ variables and $nm$ equations for the $P_i$
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In total: $2nm$ variables and $2nm$ equations of low degree over $k$.

$\implies$ We can expect that the system has 0-dimensional solution set.

(Or maybe not?)
Solving the systems

There are algorithms to determine all $k$-rational isolated solutions of multivariate systems. (Details omitted.)

A point of an algebraic set / scheme is called isolated if it is equal to its connected component.

The expected running time is $e^{O(nm)} \cdot \log(q)^{O(1)}$. (Again details omitted.)

Assume that for varying $C \in E(K)$, “most” $k$-rational solutions are isolated. Then the expected running time for the relation generation is

$$m! \cdot e^{O(nm)} \cdot q^c.$$
Solving the systems

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Assume that for varying \( C \in E(K) \), “most” \( k \)-rational solutions are isolated. Then the expected running time for the relation generation is

\[
e^{O(nm + \frac{n}{m} \cdot \log(q))}.
\]
The heuristic running time

We have

$$e^{O(nm + \frac{n}{m} \cdot \log(q))}.$$ 

for the relation generation and just $e^{O(\frac{n}{m} \cdot \log(q))}$ for the linear algebra.

For $m = \min(\lceil \sqrt{\log(q)} \rceil, n)$ we have

$$e^{O(\max(n \cdot \sqrt{\log(q)}, \log(q)))}.$$
Applications

Let us assume an expected running time of

\[ e^{O(\max(n \cdot \sqrt{\log(q)}, \log(q)))} \]

Then:

- For \( n \leq b \cdot \sqrt{\log(q)} \) we have \( q^{O(1)} \).
- For \( a \sqrt{\log(q)} \leq n \leq b \log(q) \) we have
  \[ e^{O((\log(q^n))^{2/3})} \].
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- For \( n \leq b \cdot \sqrt{\log(q)} \) we have \( q^{O(1)} \).
  
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we have

\[ e^{O((\log(q^n))^{2/3})} \].

\[ q = e^{\log(q)} = e^{(\log(q)^{3/2})^{2/3}} = e^{(\sqrt{\log(q)} \cdot \log(q))^{2/3}} \leq e^{(\frac{1}{a} n \log(q))^{2/3}} \]
Applications

Let us assume an expected running time of

\[ e^{O(\max(n \cdot \sqrt{\log(q)}, \log(q)))} \, . \]

Then:

- For

  \[ a \log(q) \leq n \leq b \log(q) \]

  we have

  \[ e^{O((\log(q^n))^{3/4})} \, . \]
Let $\text{Res}_{K|k}(E)$ be the Weil restriction of $E$ w.r.t. $K|k$. This is an $n$-dimensional abelian variety over $k$ with

$$\text{Res}_{K|k}(E)(k) \simeq E(K).$$

More generally, for any $k$-scheme $S$,

$$\text{Res}_{K|k}(E)(S) \simeq E(S \times_k K).$$
Geometry

Let $\text{Res}_{K|k}(E)$ be the Weil restriction of $E$ w.r.t. $K|k$. This is an $n$-dimensional abelian variety over $k$ with

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More generally, for any $k$-scheme $S$,

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We also have $\text{Res}_{K|k}(\mathbb{A}^1_K)$ with

$$\text{Res}_{K|k}(\mathbb{A}^1_K)(k) \simeq K.$$  

We have $\text{Res}_{K|k}(\mathbb{A}^1_K)(k) \simeq \mathbb{A}^n_k$. Such an isomorphism corresponds to an isomorphism $K \simeq k^n$. 

On the discrete logarithm problem in elliptic curves – p.23/37
Let $E_a := x^{-1}(\mathbb{A}_K^1)$ be the “affine part” of $E$.

The covering $x : E_a \longrightarrow \mathbb{A}_K^1$ induces a covering

$$\text{Res}(x) : \text{Res}_{K|k}(E_a) \longrightarrow \text{Res}_{K|k}(\mathbb{A}_K^1)$$

of degree $2^n$. 

On the discrete logarithm problem in elliptic curves – p.24/37
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of degree $2^n$.

$U \leq K$ corresponds to a subgroup-variety $A$ of $\text{Res}_{K|k}(\mathbb{A}^1_K)$ with

$$A(k) = U.$$
Geometry

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of degree $2^n$.

$U \leq K$ corresponds to a subgroup-variety $A$ of $\text{Res}_{K|k}(\mathbb{A}^1_K)$ with

$$A(k) = U.$$

Likewise $U'$ corresponds to a subgroup-variety $A'$ of $A$ with

$$A'(k) = U'.$$
Geometry

Let $V$ be defined by the following diagram being Cartesian.

\[
\begin{array}{ccc}
V & \hookrightarrow & \text{Res}_{K/k}(E_a) \\
\downarrow & & \downarrow \text{Res}(x) \\
A & \hookrightarrow & \text{Res}_{K/k}(\mathbb{A}^1_K).
\end{array}
\]

We have $\mathcal{F} \simeq V(k)$.

Let $V'$ be defined similarly. Then also $\mathcal{F} \simeq V'(k)$.
The addition map \((\mathcal{F}')^\delta \times \mathcal{F}^{m-\delta} \rightarrow E(K)\) corresponds to the addition map

\[ V'(k)^\delta \times V(k)^{m-\delta}(k) \rightarrow \text{Res}_{K/k}(E)(k). \]

Let

\[ a_m : (V')^\delta \times V(k)^{m-\delta} \rightarrow \text{Res}_{K/k}(E). \]

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The addition map \((\mathcal{F}')^\delta \times \mathcal{F}^{m-\delta} \longrightarrow E(K)\) corresponds to the addition map

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Let

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be the addition map.

For \(C \in E(K) \cong \text{Res}_{K|k}(E)(k)\) we want to study the preimage of \(C\) in \((V')^\delta \times V^{m-\delta}\).
The addition map $(\mathcal{F}')^\delta \times \mathcal{F}^{m-\delta} \rightarrow E(K)$ corresponds to the addition map

$$V'(k)^\delta \times V(k)^{m-\delta}(k) \rightarrow \text{Res}_{K|k}(E)(k).$$

Let

$$a_m : (V')^\delta \times V(k)^{m-\delta} \rightarrow \text{Res}_{K|k}(E).$$

be the addition map.

For $C \in E(K) \simeq \text{Res}_{K|k}(E)(k)$ we want to study the preimage of $C$ in $(V')^\delta \times V^{m-\delta}$.

This is called the fiber at $C$. 
Let still

$$a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E).$$

Main task. Let $C \in E(K)$ be uniformly distributed. Give now a suitable lower bound on the probability that the fiber of $C$ contains an isolated $k$-rational point!
Let still
\[ a_m : (V')^\delta \times V^{m-\delta} \rightarrow \text{Res}_{K|k}(E). \]

**Main task.** Let \( C \in E(K) \) be uniformly distributed. Give now a suitable lower bound on the probability that the fiber of \( C \) contains an isolated \( k \)-rational point!

**Note:** \( \text{Res}_{K|k}(E)(k) \) and \( (V')^\delta \times V^{m-\delta} \) have dimension \( n \).
Geometry

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**Note:** \( \text{Res}_{K|k}(E)(k) \) and \( (V')^\delta \times V^{m-\delta} \) have dimension \( n \).

**Question.** Is \( a_m : (V')^\delta \times V^{m-\delta} \rightarrow \text{Res}_{K|k}(E) \) surjective?
Let still
\[ a_m : (V')^\delta \times V^{m-\delta} \rightarrow \text{Res}_{K|k}(E). \]

**Main task.** Let \( C \in E(K) \) be uniformly distributed. Give now a suitable lower bound on the probability that the fiber of \( C \) contains an isolated \( k \)-rational point!

**Note:** \( \text{Res}_{K|k}(E)(\bar{k}) \) and \( (V')^\delta \times V^{m-\delta} \) have dimension \( n \).

**Question.** Is \( a_m : (V')^\delta \times V^{m-\delta} \rightarrow \text{Res}_{K|k}(E) \) surjective on every irreducibility component of \( (V')^\delta \times V^{m-\delta} \)?
Geometry

Let still

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**Main task.** Let \( C \in E(K) \) be uniformly distributed. Give now a suitable lower bound on the probability that the fiber of \( C \) contains an isolated \( k \)-rational point!

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**Question.** Is \( a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E) \) surjective on every irreducibility component of \( (V')^\delta \times V^{m-\delta} \)?

Difficult!
Let still
\[ a_m : (V')^\delta \times V^{m-\delta} \to \text{Res}_{K|k}(E) . \]

Observation. Let \((P_1, \ldots, P_m) \in ((V')^\delta \times V^{m-\delta})(k),\)
\[ C := P_1 + \cdots + P_m. \]
Equivalent are:

- \((P_1, \ldots, P_m)\) is isolated and reduced in the fiber of \(C\).
- \(a_m\) is unramified at \((P_1, \ldots, P_m)\).
- \((a_m)_* : T_{(P_1, \ldots, P_m)}((V')^\delta \times V^{m-\delta}) \to T_C(\text{Res}_{K|k}(E))\) is injective.

Moreover, “unramifiedness” is an open property ...
New algorithm

Choose a decomposition

\[ K = \bigoplus_{i=1}^{m} U_i. \]

Let

\[ F_i := \{ P \in E(K) \mid x(P) \in U_i \} \]

and

\[ F := \bigcup_{i=1}^{m} F_i. \]
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and

\[ F := \bigcup_{i=1}^{m} F_i. \]

We want to find relations of the form

\[ P_1 + \cdots + P_m = C \quad \text{with} \quad P_i \in F_i. \]
New algorithm

The decomposition

\[ K = \bigoplus_{i=1}^{m} U_i \]

corresponds to a decomposition

\[ \mathbb{A}^n_k \cong \text{Res}_{K/k}(\mathbb{A}^1_K) = \bigoplus_{i=1}^{m} A_i. \]

Let \( V_i \) be as \( V \) above. We thus have

\[ V_i(k) \cong \mathcal{F}_i. \]
New algorithm

Let $0 \in \mathbb{A}^1(K)$ be unramified and split under $x : E_a \to \mathbb{A}^1_K$; let $P_0 \in E(K)$ be a preimage.

Now $\text{Res}(x) : \text{Res}_{k}(E_a) \to \text{Res}_{k}(\mathbb{A}^1_K)$ is unramified at $P_0 \in \text{Res}_{k}(E)(k)$. 
Let $0 \in \mathbb{A}^1(K)$ be unramified and split under $x : E_a \rightarrow \mathbb{A}^1_K$; let $P_0 \in E(K)$ be a preimage.

Now $\text{Res}(x) : \text{Res}_K|_k(E_a) \rightarrow \text{Res}_K|_k(\mathbb{A}^1_K)$ is unramified at $P_0 \in \text{Res}_K|_k(E)(k)$.

We want to study the fibers of the map

$$a_m : V_1 \times \cdots \times V_m \rightarrow \text{Res}_K|_k(E).$$
New algorithm

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We want to study the fibers of the map

$$a_m : V_1 \times \cdots \times V_m \rightarrow \text{Res}_{K|k}(E).$$

Claim. This map is unramified at $(P_0, \ldots, P_0) \in (V_1 \times \cdots \times V_m)(k)$.

$\implies$ If the $V_i$ are irreducible, the map is generically unramified, thus generically quasi-finite.
Proof of claim.

We have

$$a_m : V_1 \times \cdots \times V_m \to \text{Res}_{K|k}(E).$$

This morphism is unramified at $(P_0, \ldots, P_0)$ if and only if the map

$$(a_m)_* : T_{(P_0, \ldots, P_0)}(V_1 \times \cdots \times V_m) \to T_{mP_0}(\text{Res}_{K|k}(E))$$

is injective.
New algorithm

We have

\[
\begin{align*}
T_{(P_0, \ldots, P_0)}(V_1 \times \cdots \times V_m) & \xrightarrow{(a_m)_*} T_{mP_0}(\text{Res}_K|_k(E)) \\
T_{P_0}(V_1) \times \cdots \times T_{P_0}(V_m) & \xrightarrow{\sum} T_{P_0}(\text{Res}_K|_k(E)) \\
T_0(A_1) \times \cdots \times T_0(A_m) & \xrightarrow{\sum} T_0(\text{Res}_K|_k(A^1_K)) \\
U_1 \times \cdots \times U_m & \xrightarrow{\sum} K.
\end{align*}
\]
New algorithm

Let now \((P_1, \ldots, P_m) \in V_1(k) \times \cdots \times V_m(k)\). Then the map

\[
a_m : V_1 \times \cdots \times V_m \longrightarrow \text{Res}_{K|k}(E)
\]

is unramified at \((P_1, \ldots, P_m)\) if and only if we have

\[
T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^{m} (\tau_{(P_0-P_i)})(T_{P_i}(V_i))
\]

This can be studied explicitly.
New algorithm

Let now \((P_1, \ldots, P_m) \in V_1(k) \times \cdots \times V_m(k)\). Then the map

\[
a_m : V_1 \times \cdots \times V_m \rightarrow \text{Res}_{K|k}(E)
\]

is unramified at \((P_1, \ldots, P_m)\) if and only if we have

\[
T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^{m}(\tau(P_0-P_i) \ast (TP_i(V_i))).
\]

This can be studied explicitly.

Let \(\text{char}(k)\) be odd. We have the holomorphic differential \(\frac{dx}{y}\) and the holomorphic tangent vector field \(yt_x\).

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New algorithm

Let now \((P_1, \ldots, P_m) \in V_1(k) \times \cdots \times V_m(k)\). Then the map

\[
a_m : V_1 \times \cdots \times V_m \rightarrow \text{Res}_{K|k}(E)
\]

is unramified at \((P_1, \ldots, P_m)\) if and only if we have

\[
TP_0 (\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^{m} (\tau(P_0-P_i))^*(TP_i(V_i))
\]

This can be studied explicitly.

Let \(\text{char}(k)\) be even, \(E\) non-supersingular. We have \(\frac{dx}{x}\) and \(xt\).
New algorithm

Let now \((P_1, \ldots, P_m) \in V_1(k) \times \cdots \times V_m(k)\). Then the map

\[ a_m : V_1 \times \cdots \times V_m \to \text{Res}_{K|k}(E) \]

is unramified at \((P_1, \ldots, P_m)\) if and only if we have

\[ T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^m (\tau_{(P_0-P_i)})(TP_i(V_i)) . \]

This can be studied explicitly.

Let \(\text{char}(k)\) be even and \(E\) supersingular. We have \(dx\) and \(t_x\).
And some conditions

Additionally, we have some conditions:

- $\#V_i(k)$ should have at least $\frac{1}{4} \cdot q^{\dim(V_i)}$ elements.
- For odd characteristic, the $V_i$ have to be irreducible.
The results

**Theorem.** The discrete logarithm problem in the groups of rational points of elliptic curves over fields $\mathbb{F}_{q^n}$ can be solved in an expected time of

$$e^{O\left(\max(\log(q), n \cdot \log(q)^{1/2}, n^{3/2})\right)}.$$ 

Under the condition that $q$ is even it can be solved in an expected time of

$$e^{O\left(\max(\log(q), n \cdot \log(q)^{1/2}, n \cdot \log(n)^{1/2})\right)}.$$
My works

- On the discrete logarithm problem in elliptic curves. Compositio Mathematica No. 147, 2011
- On the discrete logarithm problem in elliptic curves II. Submitted, 2011