

On the discrete logarithm problem in elliptic curves

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Some history

At ECC 2004 in Bochum, Pierrick Gaudry presented an index calculus algorithm for the ECDLP over extension fields:

Heuristic claim Let $n \in \mathbb{N}$, $n \geq 2$ be fixed. Then the ECDLP over fields of the form \mathbb{F}_{q^n} can be solved in an expected time of

$$O(q^{2 - \frac{2}{n}}).$$

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On the next day, I claimed:

Some history

Claim. There exists a randomized algorithm which takes as input a tuple $(q, n, E/\mathbb{F}_{q^n}, A, B)$, where q is a prime power, n a natural number, E/\mathbb{F}_{q^n} an elliptic curve and $A, B \in E(\mathbb{F}_{q^n})$ with $B \in \langle A \rangle$, which computes the DLP with respect to A and B and has the following property:

Let us fix $a, b \in \mathbb{R}$ with $0 < a < b$ and let us consider all instances with

$$a \log_2(q) \leq n \leq b \log_2(q).$$

Then restricted to these instances, the algorithm has an expected running time of

$$O\left(2^{D \cdot (n \cdot \log_2(q))^{3/4}}\right) \quad \text{for } D = \frac{4b + \epsilon}{a^{3/4}}.$$

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And I continued ...

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Please note.

1. I do not have a complete proof of this statement.
2. The algorithm is not practical.

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For fixed $a, b > 0$ and instances with

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- The algorithm is still not practical.

A preliminary algorithm

Let an instance E/\mathbb{F}_{q^n} , A, B be given, E in Weierstraß-Form.

Let us for simplicity assume that $\#E(\mathbb{F}_{q^n})$ is prime.

Let $k := \mathbb{F}_q$, $K := \mathbb{F}_{q^n}$, and let $x : E \longrightarrow \mathbb{P}_K^1$ be as usual.

A preliminary algorithm

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2. Determine some $m \leq n$ and $c \leq n$.
3. Choose some c -dimensional k -vector subspace U of K .
4. Define a so-called *factor base*

$$\mathcal{F} := \{P \in E(K) \mid x(P) \in U\}$$

Let $\mathcal{F} = \{F_1, \dots, F_k\}$.

A preliminary algorithm

5. For $i = 1, \dots, k + 1$ do

Repeat

Choose $\alpha_i, \beta_i \in \mathbb{Z}/N\mathbb{Z}$ uniformly randomly and try to determine a *relation*

$$P_1 + \dots + P_m = \alpha_i A + \beta_i B$$

with $P_1, \dots, P_m \in \mathcal{F}$.

Until this was successful.

Rewrite the relation as

$$\sum_{j=1}^k r_{i,j} F_j = \alpha_i A + \beta_i B .$$

A preliminary algorithm

6. Determine some $\underline{\gamma} \in (\mathbb{Z}/N\mathbb{Z})^{k+1} : \underline{\gamma}R = 0, \underline{\gamma} \neq \underline{0}$.

We have

$$\left(\sum_i \gamma_i \alpha_i\right)a + \left(\sum_i \gamma_i \beta_i\right)b = 0$$

and thus

$$b = \frac{-\sum_i \gamma_i \alpha_i}{\sum_i \gamma_i \beta_i} a .$$

Relation generation

Given $C (= \alpha A + \beta B) \in E(K)$, we want to find a relation

$$P_1 + \cdots + P_m = C$$

with $P_1, \dots, P_m \in \mathcal{F}$.

For this we try to solve systems of multivariate polynomial equations over k .

Relation generation

Idea. For $P_1, \dots, P_m \in E(\overline{K})$, the condition $P_1 + \dots + P_m = C$ can be expressed algebraically over K .

We try to find relations by solving systems of polynomial equations over k .

- The space of tuples $(P_1, \dots, P_m) \in \mathcal{F}^m$ has mc degrees of freedom over k .
- The space of points $C \in E(K)$ has n degrees of freedom over k .

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\implies Let $\delta := mc - n$. Then for fixed C the relations / solutions $(P_1, \dots, P_m) \in \mathcal{F}^m$ with $P_1 + \dots + P_m = C$ vary in a δ -dimensional space over k .

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We want that $\delta = 0 \dots$

A new preliminary algorithm

1. Determine $N := \#E(K)$.
2. Determine some $m \leq n$, let $c := \lceil \frac{n}{m} \rceil$ and $\delta := mc - n$.
We thus have $n = mc - \delta = (m - \delta) \cdot c + \delta \cdot (c - 1)$.
3. Choose some c -dimensional k -vector subspace U of K and some $c - 1$ -dimensional k -vector subspace U' of U .
4. Define a *factor base*

$$\mathcal{F} := \{P \in E(K) \mid x(P) \in U\}$$

and also

$$\mathcal{F}' := \{P \in E(K) \mid x(P) \in U'\}.$$

Let $\mathcal{F} = \{F_1, F_2, \dots, F_k\}$.

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with $P_1, \dots, P_\delta \in \mathcal{F}'$, $P_{\delta+1}, \dots, P_m \in \mathcal{F}$.

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and thus

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Decomposition

We need a procedure to compute relations or “decompositions”.

Input. $C \in E(K)$.

Output. A relation

$$P_1 + \cdots + P_m = C$$

with

$$P_1, \dots, P_\delta \in \mathcal{F}', P_{\delta+1}, \dots, P_m \in \mathcal{F},$$

that is,

$$x(P_1), \dots, x(P_\delta) \in U', x(P_{\delta+1}), \dots, x(P_m) \in U.$$

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- $\exists f \in K(E)^* : (f) = (P_1) + \dots + (P_m) + (-C) - (m + 1) \cdot (O)$.
- $\exists f \in L((m + 1) \cdot O - (-C)) :$
 $(f) = (P_1) + \dots + (P_m) + (-C) - (m + 1) \cdot (O)$.

Decomposition

Let $P_1, \dots, P_m \in E(K)$. Let P_1, \dots, P_m, C, O be distinct.

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Now: Choose a basis of $L((m+1) \cdot O - (-C))$, expand this over k , restrict $x(P_i)$ to U or to U' ...

Decompositions

Let $C, P_1, \dots, P_m \in E(K)$. Let P_1, \dots, P_m, C, O be distinct.
Let b_1, \dots, b_m be a basis of $L((m+1) \cdot O - (-C))$.

Equivalent are:

- $P_1 + \dots + P_m = C$
- $\exists \alpha_1, \dots, \alpha_m \in K : \forall i = 1, \dots, m : (\sum_{\ell} \alpha_{\ell} b_{\ell})(P_i) = 0$

For varying P_1, \dots, P_m , we have

- $2m$ variables for the P_i and m equations of degree 3
- $m - 1$ variables for the $\alpha_1, \dots, \alpha_{m-1}$ and m equations of low degree.

Over k , we have

- $nm + n$ variables and nm equations for the P_i
- $nm - n$ variables and nm additional equations.

Solving the systems

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In total: $2nm$ variables and $2nm$ equations of low degree over k .

⇒ We can expect that the system has 0-dimensional solution set.

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(Or maybe not?)

Solving the systems

There are algorithms to determine all k -rational *isolated* solutions of multivariate systems. (Details omitted.)

A point of an algebraic set / scheme is called *isolated* if it is equal to its connected component.

The expected running time is $e^{O(nm)} \cdot \log(q)^{O(1)}$. (Again details omitted.)

Assume that for varying $C \in E(K)$, “most” k -rational solutions are isolated. Then the expected running time for the relation generation is

$$m! \cdot e^{O(nm)} \cdot q^c .$$

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Assume that for varying $C \in E(K)$, “most” k -rational solutions are isolated. Then the expected running time for the relation generation is

$$e^{O(nm + \frac{n}{m} \cdot \log(q))} .$$

The heuristic running time

We have

$$e^{O(nm + \frac{n}{m} \cdot \log(q))} .$$

for the relation generation and just $e^{O(\frac{n}{m} \cdot \log(q))}$ for the linear algebra.

For $m = \min(\lceil \sqrt{\log(q)} \rceil, n)$ we have

$$e^{O(\max(n \cdot \sqrt{\log(q)}, \log(q)))} .$$

Applications

Let us assume an expected running time of

$$e^{O(\max(n \cdot \sqrt{\log(q)}, \log(q)))} .$$

Then:

- For $n \leq b \cdot \sqrt{\log(q)}$ we have $q^{O(1)}$.

For

$$a\sqrt{\log(q)} \leq n \leq b \log(q)$$

we have

$$e^{O((\log(q^n))^{2/3})} .$$

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$$q = e^{\log(q)} = e^{(\log(q)^{3/2})^{2/3}} = e^{(\sqrt{\log(q)} \cdot \log(q))^{2/3}} \leq e^{(\frac{1}{a} n \log(q))^{2/3}}$$

Applications

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Then:

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we have

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Geometry

Let $\text{Res}_{K|k}(E)$ be the Weil restriction of E w.r.t. $K|k$. This is an n -dimensional abelian variety over k with

$$\text{Res}_{K|k}(E)(k) \simeq E(K) .$$

More generally, for any k -scheme S ,

$$\text{Res}_{K|k}(E)(S) \simeq E(S \times_k K) .$$

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We also have $\text{Res}_{K|k}(\mathbb{A}_K^1)$ with

$$\text{Res}_{K|k}(\mathbb{A}_K^1)(k) \simeq K .$$

We have $\text{Res}_{K|k}(\mathbb{A}_K^1)(k) \simeq \mathbb{A}_k^n$. Such an isomorphism corresponds to an isomorphism $K \simeq k^n$.

Geometry

Let $E_a := x^{-1}(\mathbb{A}_K^1)$ be the “affine part” of E .

The covering $x : E_a \longrightarrow \mathbb{A}_K^1$ induces a covering

$$\text{Res}(x) : \text{Res}_{K|k}(E_a) \longrightarrow \text{Res}_{K|k}(\mathbb{A}_K^1)$$

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Likewise U' corresponds to a subgroup-variety A' of A with

$$A'(k) = U' .$$

Geometry

Let V be defined by the following diagram being Cartesian.

$$\begin{array}{ccc} V \hookrightarrow & \text{Res}_{K|k}(E_a) & \\ \downarrow & & \downarrow \text{Res}(x) \\ A \hookrightarrow & \text{Res}_{K|k}(\mathbb{A}_K^1) & . \end{array}$$

We have $\mathcal{F} \simeq V(k)$.

Let V' be defined similarly. Then also $\mathcal{F} \simeq V'(k)$.

Geometry

The addition map $(\mathcal{F}')^\delta \times \mathcal{F}^{m-\delta} \longrightarrow E(K)$ corresponds to the addition map

$$V'(k)^\delta \times V(k)^{m-\delta}(k) \longrightarrow \text{Res}_{K|k}(E)(k) .$$

Let

$$a_m : (V')^\delta \times V(k)^{m-\delta} \longrightarrow \text{Res}_{K|k}(E) .$$

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For $C \in E(K) \simeq \text{Res}_{K|k}(E)(k)$ we want to study the preimage of C in $(V')^\delta \times V^{m-\delta}$.

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For $C \in E(K) \simeq \text{Res}_{K|k}(E)(k)$ we want to study the preimage of C in $(V')^\delta \times V^{m-\delta}$.

This is called the *fiber* at C .

Geometry

Let still

$$a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E) .$$

Main task. Let $C \in E(K)$ be uniformly distributed. Give now a suitable lower bound on the probability that the fiber of C contains an isolated k -rational point!

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Note: $\text{Res}_{K|k}(E)(k)$ and $(V')^\delta \times V^{m-\delta}$ have dimension n .

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Question. Is $a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E)$ surjective?

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Question. Is $a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E)$ surjective on every irreducibility component of $(V')^\delta \times V^{m-\delta}$?

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Difficult!

Geometry

Let still

$$a_m : (V')^\delta \times V^{m-\delta} \longrightarrow \text{Res}_{K|k}(E) .$$

Observation. Let $(P_1, \dots, P_m) \in ((V')^\delta \times V^{m-\delta})(k)$,
 $C := P_1 + \dots + P_m$.

Equivalent are:

- (P_1, \dots, P_m) is isolated and reduced in the fiber of C .
- a_m is unramified at (P_1, \dots, P_m) .
- $(a_m)_* : T_{(P_1, \dots, P_m)}((V')^\delta \times V^{m-\delta}) \longrightarrow T_C(\text{Res}_{K|k}(E))$ is injective.

Moreover, “unramifiedness” is an open property ...

New algorithm

Choose a decomposition

$$K = \bigoplus_{i=1}^m U_i .$$

Let

$$\mathcal{F}_i := \{P \in E(K) \mid x(P) \in U_i\}$$

and

$$\mathcal{F} := \bigcup_{i=1}^m \mathcal{F}_i .$$

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We want to find relations of the form

$$P_1 + \cdots + P_m = C \text{ with } P_i \in \mathcal{F}_i .$$

New algorithm

The decomposition

$$K = \bigoplus_{i=1}^m U_i$$

corresponds to a decomposition

$$\mathbb{A}_k^n \simeq \text{Res}_{K|k}(\mathbb{A}_K^1) = \bigoplus_{i=1}^m A_i .$$

Let V_i be as V above. We thus have

$$V_i(k) \simeq \mathcal{F}_i .$$

New algorithm

Let $0 \in \mathbb{A}^1(K)$ be unramified and split under $x : E_a \longrightarrow \mathbb{A}^1_K$;
let $P_0 \in E(K)$ be a preimage.

Now $\text{Res}(x) : \text{Res}_{K|k}(E_a) \longrightarrow \text{Res}_{K|k}(\mathbb{A}^1_K)$ is unramified at
 $P_0 \in \text{Res}_{K|k}(E)(k)$.

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We want to study the fibers of the map

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We want to study the fibers of the map

$$a_m : V_1 \times \cdots \times V_m \longrightarrow \text{Res}_{K|k}(E) .$$

Claim. This map is unramified at
 $(P_0, \dots, P_0) \in (V_1 \times \cdots \times V_m)(k)$.

\implies If the V_i are irreducible, the map is
generically unramified, thus generically quasi-finite.

New algorithm

Proof of claim.

We have

$$a_m : V_1 \times \cdots \times V_m \longrightarrow \text{Res}_{K|k}(E) .$$

This morphism is unramified at (P_0, \dots, P_0) if and only if the map

$$(a_m)_* : T_{(P_0, \dots, P_0)}(V_1 \times \cdots \times V_m) \longrightarrow T_{mP_0}(\text{Res}_{K|k}(E))$$

is injective.

New algorithm

We have

$$\begin{array}{ccc}
 T_{(P_0, \dots, P_0)}(V_1 \times \dots \times V_m) & \xrightarrow{(a_m)_*} & T_{mP_0}(\text{Res}_{K|k}(E)) \\
 \updownarrow & & \up (\tau_{((m-1)P_0)})_* \\
 T_{P_0}(V_1) \times \dots \times T_{P_0}(V_m) & \xrightarrow{\Sigma} & T_{P_0}(\text{Res}_{K|k}(E)) \\
 \downarrow & & \downarrow \text{Res}(x)_* \\
 T_0(A_1) \times \dots \times T_0(A_m) & \xrightarrow{\Sigma} & T_0(\text{Res}_{K|k}(\mathbb{A}_K^1)) \\
 \parallel & & \parallel \\
 U_1 \times \dots \times U_m & \xrightarrow{\Sigma} & K .
 \end{array}$$

New algorithm

Let now $(P_1, \dots, P_m) \in V_1(k) \times \dots \times V_m(k)$. Then the map

$$a_m : V_1 \times \dots \times V_m \longrightarrow \text{Res}_{K|k}(E)$$

is unramified at (P_1, \dots, P_m) if and only if we have

$$T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^m (\tau_{(P_0 - P_i)})_* (T_{P_i}(V_i)) .$$

This can be studied explicitly.

New algorithm

Let now $(P_1, \dots, P_m) \in V_1(k) \times \dots \times V_m(k)$. Then the map

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This can be studied explicitly.

- Let $\text{char}(k)$ be odd. We have the holomorphic differential $\frac{dx}{y}$ and the holomorphic tangent vector field yt_x .

New algorithm

Let now $(P_1, \dots, P_m) \in V_1(k) \times \dots \times V_m(k)$. Then the map

$$a_m : V_1 \times \dots \times V_m \longrightarrow \text{Res}_{K|k}(E)$$

is unramified at (P_1, \dots, P_m) if and only if we have

$$T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^m (\tau_{(P_0 - P_i)})_* (T_{P_i}(V_i)) .$$

This can be studied explicitly.

- Let $\text{char}(k)$ be even, E non-supersingular. We have $\frac{dx}{x}$ and xt_x .

New algorithm

Let now $(P_1, \dots, P_m) \in V_1(k) \times \dots \times V_m(k)$. Then the map

$$a_m : V_1 \times \dots \times V_m \longrightarrow \text{Res}_{K|k}(E)$$

is unramified at (P_1, \dots, P_m) if and only if we have

$$T_{P_0}(\text{Res}_{K|k}(E)) = \bigoplus_{i=1}^m (\tau_{(P_0 - P_i)})_* (T_{P_i}(V_i)) .$$

This can be studied explicitly.

- Let $\text{char}(k)$ be even and E supersingular. We have dx and t_x .

And some conditions

Additionally, we have some conditions:

- $\#V_i(k)$ should have at least $\frac{1}{4} \cdot q^{\dim(V_i)}$ elements.
- For odd characteristic, the V_i have to be irreducible.

The results

Theorem. The discrete logarithm problem in the groups of rational points of elliptic curves over fields \mathbb{F}_{q^n} can be solved in an expected time of

$$e^{O(\max(\log(q), n \cdot \log(q)^{1/2}, n^{3/2}))} .$$

Under the condition that q is even it can be solved in an expected time of

$$e^{O(\max(\log(q), n \cdot \log(q)^{1/2}, n \cdot \log(n)^{1/2}))} .$$

My works

- Habilitation thesis: On arithmetic and the discrete logarithm problem in class groups of curves, 2008
- On the discrete logarithm problem in elliptic curves. Compositio Mathematica No. 147, 2011
- On the discrete logarithm problem in elliptic curves II. Submitted, 2011