Elliptic Periods & Applications

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Motivation

- Let R be a (commutative and unitary) ring, the algebra $S = R[x]/(x^d - \alpha)$ has shown to be (algorithmically) very useful:
  - Low complexity normal basis [GL92];
  - Primality proving [AKS04];
  - Discrete Logarithm computations in Finite Fields [JL06];
  - Fast polynomial factorization and composition [KU08].
- But, often, there is no primitive d-th root of unity in R (and embedding the ring R into an auxiliary extension R' yields important losses of efficiency).
- Idea: substitute to S one elliptic curve E defined on R, having a point $T \in E(R)$ of exact order d.

Joint works with J.-M. Couveignes, C. Dunand, T. Ezome.

Outline

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Classical Method

A classical approach:

- Choosing a random polynomial of degree d.
- Testing for its irreducibility.

Complexity:

- The probability that a polynomial of degree d be irreducible is at least $1/(2d)$ [LN83, Ex. 3.26 and 3.27, page 142]
- Ben-Or's irreducibility test [BO81], this test has average complexity $(\log q)^{1+o(1)} \times d^{1+o(1)}$ elementary operations

A total of $(\log q)^{2+o(1)} \times d^{2+o(1)}$ elementary operations.

Another approach [CL09b]

Difficult to improve things as long as we use an irreducibility test.
We are thus driven to consider very particular polynomials.

Adleman and Lenstra [AL86] construct such irreducible polynomials (thanks to Gauss periods),

- with (now) complexity quasi-linear in d,
- but only when $d = \ell^\delta$ with ℓ a prime divisor of $p(q-1)$.

We mimic their construction using isogenies between elliptic curves,

- with still complexity quasi-linear in d,
- but $d = \ell^\delta$ is coprime to $p\,(q-1)$.

A total complexity of $d^{1+o(1)} \times (\log q)^{5+o(1)}$.

Artin-Schreier towers : $d = p^\delta$ [LdS08]

For every $k \in \mathbb{N}^*$, le $\mathcal{A}_k \subset \overline{\mathbb{F}}_p$ be the subset of a's in $\overline{\mathbb{F}}_p$ s.t.

1. a generates $\mathbb{F}_{p^k}$ over $\mathbb{F}_p$, *i.e.* $\mathbb{F}_p(a) = \mathbb{F}_{p^k}$,
2. a has non-zero absolute trace, *i.e.* $\mathrm{Tr}\, a \neq 0$,
3. a^{-1} has non-zero absolute trace, *i.e.* $\mathrm{Tr}\, a^{-1} \neq 0$.

Especially, $\mathcal{A}_1 = \mathbb{F}_p^*$.

Let now I be the map

$$I : \begin{array}{rcl} \overline{\mathbb{F}}_p \setminus \mathbb{F}_p & \to & \overline{\mathbb{F}}_p \setminus \{0\} \\ X & \mapsto & (X^p - 1)/(X + X^2 + \cdots + X^{p-1}) \end{array}$$

We check that

- $I^{-1}(\mathcal{A}_k) \subset \mathcal{A}_{pk}$,
- $I^{-\delta}(1)$ is a degree p^δ irreducible divisor over $\mathbb{F}_p$.

Examples

If $p = 2$, $d = 2$:

- Compute $I(x) = \dfrac{x^2 + 1}{x}$;
- $f(x) = x^2 + 1 - x$.

If $p = 2$, $d = 4$:

- Compute $(I \circ I)(x) = \dfrac{x^4 + x^2 + 1}{x^3 + x}$;
- $f(x) = x^4 + x^2 + 1 - (x^3 + x)$.

Both are irreducible polynomials in $\mathbb{F}_2[x]$.

Radicial extensions : $d = \ell^\delta$ with $\ell | p - 1$

If $\ell = 2$, we ask that $4 | p - 1$.

First, look for a generator a of the ℓ-Sylow subgroup of $\mathbb{F}_p^*$.

- Pick random α in $\mathbb{F}_p^*$ until $a = \alpha^{(p-1)/\ell^e} \neq 1$.
- The probability of success is about 1.

Then the polynomial $f(x) = x^d - a$ is irreducible in $\mathbb{F}_p[x]$.

Proof.

- The $\ell^{\delta+e}$-torsion $\mathbf{G}_m[\ell^{\delta+e}]$ of $\mathbf{G}_m$ is isomorphic to $(\mathbb{Z}/\ell^{\delta+e}\mathbb{Z}, +)$
- The Frobenius $\varphi_q : \mathbf{G}_m \to \mathbf{G}_m$ acts on it as mult. by q.
- The order of $q = 1 + \ell'\ell^e$ in $(\mathbb{Z}/\ell^{e+\delta}\mathbb{Z})^*$ is $\ell^\delta = d$.
- So the Frobenius Φ_q acts transitively on the roots of $f(x)$.

□

Example

We take $p = 5$, $\ell = 2$, $\delta = 3$ and $d = 8$.

- We check that 4 divides $p - 1$.
- In particular $e = 2$ and $\ell' = 1$.
- The class $a = 2 \bmod 5$ generates the 2-Sylow subgroup of $(\mathbb{Z}/5\mathbb{Z})^*$.

  ($2^4 = 1 \bmod 5$ and $2^2 = -1 \bmod 5$).
- We set $f(x) = x^8 - 2$.

Residue fields of divisors on elliptic curves

Let E be an elliptic curve defined over $\mathbb{F}_p$.

- Assume $E(\mathbb{F}_p)$ contains a cyclic subgroup $\mathcal{T}$ of order d.
- Let $I : E \to E'$ be the degree d cyclic isogeny with kernel $\mathcal{T}$
- Take a in $E'(\mathbb{F}_p)$ of order d.
- Consider the fibre $I^{-1}(a) = \sum_{T \in \mathcal{T}}[b + t]$.

$$\begin{array}{lcccl}
I^{-1}(a) = \sum_{T \in \mathcal{T}}[b + t] \subset & E(\mathbb{F}_{p^d}) & & & \\
 & \Big| \, d & \searrow^{I} & & \\
\mathcal{T} = \langle t \rangle \subset & E(\mathbb{F}_p) & \xrightarrow{I} & E'(\mathbb{F}_p) \ni a &
\end{array}$$

Irreducibility conditions

We factor $p+1-t = dd'$ where d' is coprime to d.

There exists two integers λ and μ such that

$$\begin{array}{rcl} X^2 - tX + q & = & (X-\lambda)(X-\mu) \bmod d^2\,, \\ \lambda = 1 \bmod d, & & \mu = q \bmod d\,. \end{array}$$

Remember $I(b) = a$, then b is a d^2-torsion point, and

$$\varphi(b) = \lambda b \text{ (where } \varphi \text{ is the Frobenius map)}\,.$$

- The order of $\lambda = 1 + d\lambda' \bmod d^2$ is equal to d.
- Thus the Galois orbit of b has cardinality d
- And the d geometric points $b+t$ above a are defined on a degree d extension $\mathbb{F}_{q^d}$ of $\mathbb{F}_p$ (and permuted by Galois action).

$\mathbb{F}_{q^d}$ is the residue extension of $\mathbb{F}_p(E)$ at $\mathcal{P} = \sum_{T\in\mathcal{T}}[b+T]$.

Example

We take $p = 7$, $q = 7$ and $d = 5$.

The elliptic curve $E/\mathbb{F}_7 \; : y^2 = x^3 + x + 4$ has got 10 $\mathbb{F}_7$-rational points.

The point $t = (6, 4)$ has order $\ell = 5$ and

$$\langle t \rangle = \{O_E,\ (6,4),\ (4,4),\ (4,3),\ (6,3)\}\ .$$

The quotient by $\langle t \rangle$ isogenous curve E', given by Vélu's formulae, is

$$E' \; : y'^2 = x'^3 + 3\,x' + 4.$$

where, x' in terms of x alone,

$$x' = x + \frac{x+2}{(x+1)^2} + \frac{1}{(x+3)^2} = \frac{x^5 + x^4 + 2\,x^3 + 5\,x^2 + 4\,x + 5}{(x+3)^2\,(x+1)^2}\ .$$

We choose $a = (1, 1)$ in $E'(\mathbb{F}_7)$ and finally obtain,

$$\begin{aligned} f_a(x) &= x^5 + x^4 + 2\,x^3 + 5\,x^2 + 4\,x + 5 - 1\ \ (x+3)^2\,(x+1)^2 \\ &= x^5 + x^3 + 4\,x^2 + x + 3\,. \end{aligned}$$

Irreducible polynomials of degree $d = \ell^\delta$

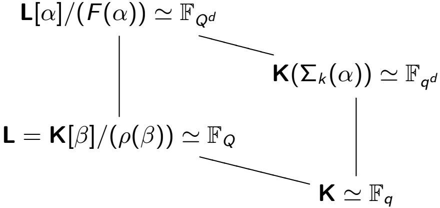
Algorithm for $4\ell \leqslant q^{\frac{1}{4}}$ and any δ :

- Pick a random elliptic curve E over $\mathbf{K}$ and compute its cardinality using Schoof's algorithm ($(\log q)^{5+o(1)}$ elem. ops).
- Repeat until the cardinality of E is divisible by ℓ (by a result of Howe, the average number of trials is $O(\ell)$).
- Compute a chain of δ quotient isogenies of degree ℓ from E with Vélu's formulas ($d^{1+o(1)} \times \ell^{1+o(1)} \times (\log q)^{2+o(1)}$ elem. ops).
- Compose these isogenies with Kedlaya-Umans' algorithm ($d^{1+o(1)} \times (\log q)^{1+o(1)}$ elem. ops).

A total of $\ell \times (\log q)^{5+o(1)} + d^{1+o(1)} \times (\log q)^{2+o(1)}$ elem. ops.

Base change

Now, assume $4\ell > q^{\frac{1}{4}}$, we have to base change to aux. extensions.



1. Find a degree $r \simeq (\log \ell)$ irreducible polynomial $\rho(\beta) \in \mathbf{K}[\beta]$ (negligible cost);
2. Obtain an irreducible polynomial $F(x)$ of degree d in $\tilde{\mathbf{L}}[x]$, in time $(\log q)^{5+o(1)} d^{1+o(1)}$ elem. ops;
3. There exists a symmetric function Σ_k such that the polynomial

$$f(x) = \prod_{0 \leqslant l < d} (x - \Phi_q^l(\Sigma_k(\alpha))) \in \mathbf{K}[x] \text{ is irreducible of degree } d.$$

Some technicalities

Three questions to be considered.

1. How to compute $\Sigma_k(\alpha)$ and its conjugates ?
   - $\alpha = x(b)$ where b is a geometric point of order $\ell^{e+\delta}$ in $E(\mathbf{L})$, so

     $$\exists\lambda \text{ s.t. } \varphi_E(b) = \lambda b \qquad (\varphi_E \text{ is the degree } Q \text{ Frobenius of } E/\mathbf{L})$$

2. How to find the good integer k ?
   - Compute the conjugates of α and form the pol. with these roots.
   - $\Sigma_k(\alpha)$ generates the degree d extension of $\mathbf{K}$ iff $\Phi_q^{\ell^{\delta-1}}(\Sigma_k(\alpha)) \neq \Sigma_k(\alpha)$, that is $\Sigma_k(\Phi_q^{\ell^{\delta-1}}(\alpha)) \neq \Sigma_k(\alpha)$.

3. How to compute $f(x) \in \mathbf{K}[x]$?
   - Compute the minimal pol.of $\Sigma_k(\alpha)$, with Kedlaya-Umans algorithm.

A total of $d^{1+o(1)} \times (\log q)^{2+o(1)}$ elem. ops

Compositum

The last problem to be considered is the following.

> *Given 2 irreducible polynomials $f_1(x)$ and $f_2(x)$ with coprime degrees d_1 and d_2, construct a deg. $d_1 d_2$ irreducible polynomial.*

This is a classical result.

- Let α_1 be a root of $f_1(x)$ and α_2 be a root of $f_2(x)$, then $\alpha_1 + \alpha_2$ generates an extension of degree $d_1 d_2$ of $\mathbb{F}_q$.
- The minimal polynomial of $\alpha_1 + \alpha_2$, called *composed sum* in a work of Bostan, Flajolet, Salvy and Schost, can be computed in quasi-linear time complexity in $d_1 d_2$.

A total of $(d_1 d_2)^{1+o(1)} \times (\log q)^{1+o(1)}$ elem. ops.

(Special) Irreducible polynomials over finite fields

Theorem

There exists an algorithm that on input a finite field $\mathbb{F}_q$, and a positive integer d, returns a degree d irreducible polynomial in $\mathbb{F}_q[X]$. The algorithm requires $d^{1+o(1)} \times (\log q)^{5+o(1)}$ elementary operations.

Remarks.

- We consider very particular polynomials (derived from points on elliptic curves).
- Some special cases $\ell = 2, 3$ have to be handled in specific ways.

(Random) Irreducible polynomials over finite fields

Given a *special* irreducible polynomial $f(x)$ of degree d, one can compute a *random* irreducible polynomial $g(x)$ of degree d with only $d^{1+o(1)} \times (\log q)^{1+o(1)}$ elementary operations.

- Choose a random element a in $\mathbf{L} = \mathbf{K}[x]/(f(x))$ (generates $\mathbf{L}$ with probability greater than $1 - \frac{q}{q-1}(q^{-\frac{d}{2}} - q^{-d}) > 1/2$);
- Compute the minimal polynomial of the element a (at the expense of $d^{1+o(1)}(\log q)^{1+o(1)}$ with Kedlaya-Umans' algorithm);

Outline

Normal basis

Given a finite field $\mathbb{F}_q$, and an integer d, how can we construct $\mathbb{F}_{q^d}$ s.t. the addition, the multiplication and q^{th} power are fast operations,

at most $\tilde{O}(d \log q)$ elementary operations ?

A first remark: Since $\mathbb{F}_{q^d}$ is a $\mathbb{F}_q$-vector space of dim. d,

- it is "natural" to represent elements as vectors over $\mathbb{F}_q$,

$$\vec{\alpha} = (\alpha_i)_{i \in \mathbb{Z}/d\mathbb{Z}},$$

- and addition is obviously fast.

But how about about multiplications and Frobenius maps ?

Ingredient 1: Residue fields of divisors on elliptic curves (again)

$$I^{-1}(a) = \sum_{T \in \mathcal{T}} [b + t] \subset \quad E(\mathbb{F}_{p^d})$$

$$E(\mathbb{F}_{p^d}) \xrightarrow{\;I\;} E'(\mathbb{F}_p), \qquad E(\mathbb{F}_{p^d}) \overset{d}{\text{—}} E(\mathbb{F}_p)$$

$$\mathcal{T} = \langle t \rangle \subset \quad E(\mathbb{F}_p) \xrightarrow{\;I\;} E'(\mathbb{F}_p) \ni a$$

Again, under some mild condition, $\phi(b) - b$ is a generator of $\mathcal{T}$ and the d geometric points above a are defined on a degree d extension $\mathbb{F}_{q^d}$ of $\mathbb{F}_q$ (and permuted by Galois action).

$\mathbb{F}_{q^d}$ is the residue extension of $\mathbb{F}_q(E)$ at $\mathcal{P}$.

Ingredient 2 : simple functions

- Let $E/\mathbb{F}_q$ be an elliptic curve given by

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3 .$$

- If A, B and C are three pairwise distinct points in $E(\mathbb{F}_q)$, we define

$$\Gamma(A,B,C) = \frac{y(C-A) - y(A-B)}{x(C-A) - x(A-B)} .$$

- We define a function $u_{A,B} \in \mathbb{F}_q(E)$ by $u_{A,B}(C) = \Gamma(A,B,C)$.

  It has degree two with two simple poles, at A and B.

Elliptic Normal Basis

Coming back to the functions u_{AB}, we choose for A and B "consecutive points" in $\mathcal{T}$.

For $k \in \mathbb{Z}/d\mathbb{Z}$, we more precisely set

$$u_k = \mathfrak{a} u_{kt,(k+1)t} + \mathfrak{b}$$

($\mathfrak{a}$ and $\mathfrak{b}$, constants chosen such that $\sum u_k = 1$),

and we evaluate the u_k's at b.

Lemma (A normal basis)

The system $\Theta = (u_k(b))_{k \in \mathbb{Z}/d\mathbb{Z}}$ is a $\mathbb{F}_q$ normal basis of $\mathbb{F}_{q^d}$.

Θ is a basis

Let λ_k in $\mathbb{F}_q$ such that $\sum_{k\in\mathbb{Z}/d\mathbb{Z}} \lambda_k u_k(b) = 0$.

Let us consider the function $f = \sum_{k\in\mathbb{Z}/d\mathbb{Z}} \lambda_k u_k$.

- It cancels not only at b, but at $b+t$ with $t \in \mathcal{T}$ (because f is defined over $\mathbb{F}_q$).
- And f has d poles, the points in $\mathcal{T}$.
- Let us assume $f \neq 0$, then $(f) = (f)_0 - (f)_\infty$ with

$$(f)_0 = \sum_{t\in\mathcal{T}} [b+t] \text{ and } (f)_\infty = \sum_{t\in\mathcal{T}} [t].$$

- So, $\sum_{t\in\mathcal{T}} (b+t) - (t) = d\, b = 0_E$. This is impossible $\Rightarrow f = 0$.
- Taylor expansions at poles show that all λ_k's are equal.
- Since $\sum u_k = 1$, all λ_k's are thus null.

Θ is normal

We have

$$\begin{aligned} \phi(u_k(b)) &= u_k(\phi(b)), \\ &= u_k(b+t). \end{aligned}$$

Remember that by def. $u_k = \mathfrak{a} u_{kt,(k+1)t} + \mathfrak{b}$, and thus

$$\begin{aligned} \phi(u_k(b)) &= \mathfrak{a} u_{kt,(k+1)t}(b+t) + \mathfrak{b}, \\ &= \mathfrak{a} u_{(k-1)t,kt}(b) + \mathfrak{b}. \\ &= u_{k-1}(b). \end{aligned}$$

Ingredient 2: Relations among elliptic functions

We can prove the following identities (with Taylor expansions at poles)

$$\begin{aligned} \Gamma(A,B,C) &= \Gamma(B,C,A) = -\Gamma(B,A,C) - a_1 \\ &= -\Gamma(-A,-B,-C) - a_1 \,, \\ u_{A,B} + u_{B,C} + u_{C,A} &= \Gamma(A,B,C) - a_1 \,, \end{aligned}$$

and

$$\begin{aligned} u_{A,B}u_{A,C} &= x_A + \Gamma(A,B,C)u_{A,C} + \Gamma(A,C,B)u_{A,B} \\ &\quad + a_2 + x_A(B) + x_A(C) \,, \\ u_{A,B}^2 &= x_A + x_B - a_1 u_{A,B} + x_A(B) + a_2 \,, \end{aligned}$$

where

- $\tau_A : E \to E$ denotes the translation by A,
- and in $\mathbb{F}_q(E)$, $x_A = x \circ \tau_{-A}$ and $y_A = y \circ \tau_{-A}$.

A fast multiplication algorithm

$$\begin{aligned} u_{A,B}u_{A,C} &= x_A + \Gamma(A,B,C)u_{A,C} + \Gamma(A,C,B)u_{A,B} \\ &\quad + a_2 + x_A(B) + x_A(C)\,, \\ u_{A,B}^2 &= x_A + x_B - a_1 u_{A,B} + x_A(B) + a_2\,. \end{aligned}$$

This yields a multiplication tensor for Θ with quasi-linear complexity,

$$\vec{\alpha} \times \vec{\beta} = (\mathfrak{a}^2 \overrightarrow{\iota}) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta})) \Big) + \overrightarrow{u_R}^{(-1)} \star \Big((\vec{u}_R \star \vec{\alpha}) \diamond (\vec{u}_R \star \vec{\beta}) - (\mathfrak{a}^2 \vec{x}_R) \star \Big((\vec{\alpha} - \sigma(\vec{\alpha})) \diamond (\vec{\beta} - \sigma(\vec{\beta})) \Big) \Big)\ .$$

Notations :

- $\vec{\alpha} \star \vec{\beta}$, the convolution product $(\vec{\alpha} \star_j \vec{\beta})_j$, with $\vec{\alpha} \star_j \vec{\beta} = \sum_i \alpha_i \beta_{j-i}$.
- $\sigma(\vec{\alpha}) = (\alpha_{i-1})_i$, the cyclic shift of $\vec{\alpha}$.
- $\vec{\alpha} \diamond \vec{\beta} = (\alpha_i \beta_i)_i$, the component-wise product.

The result [CL09a]

Theorem

To every couple (q, d) with q a prime power and $d \geqslant 2$ an integer s.t. $d_q \leqslant \sqrt{q}$, one can associate a normal basis $\Theta(q, d)$ of the degree d extension of $\mathbb{F}_q$ such that the following holds:

- *There exists an algorithm that multiplies two elements given in $\Theta(q, d)$ at the expense of $\tilde{O}(d \log q)$ elementary operations.*

This can be easily extend to a result without any restriction on q and d.

Remark: Here d_q is such that

- $v_\ell(d_q) = v_\ell(d)$ if ℓ is prime to $q-1$, $v_\ell(d_q) = 0$ if $v_\ell(d) = 0$,
- $v_\ell(d_q) = \max(2v_\ell(q-1)+1, 2v_\ell(d))$ if ℓ divides both $q-1$ and d.

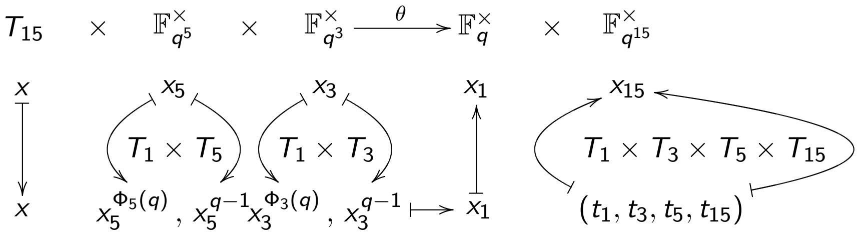
Application to Torus-based cryptography [DL09]

We have $q^n - 1 = \prod_{d \mid n} \Phi_d(q)$, and thus $\mathbb{F}_q^\times \simeq \prod_{d \mid n} T_d(\mathbb{F}_q)$.

$T_n(\mathbb{F}_q) \cong \{x \in \mathbb{F}_{q^n}^\times : x^{\Phi_n(q)} = 1\}$ is an alg. variety of dimension $\varphi(n)$.

Often, no known rational parameterization of $T_n(\mathbb{F}_q)$ with $\varphi(n)$-tuples.

Elliptic basis may yield efficient variants of a nice workaround due to van Dijk and Woodruff.



Conclusion

- We made use of torsion points on elliptic curves for finite field algorithms :
  - irreducible polynomials,
  - normal basis,
  - torus-based cryptography
  - discrete logarithms (in some very particular cases)
- It seems useful in other situations,
  - over the integers, with an elliptic AKS primality criterion,
  - over the p-adics, for counting points on curves.

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